Sinusoids and their Spectrum representation

Signals and codes (SK)

Department of Transport Telematics
Faculty of Transportation Sciences, CTU in Prague

Lecture 2

Lecture goal and content

Goal

- Be able to find spectral representation of sinusoidal signals and understand what does it mean.

Content

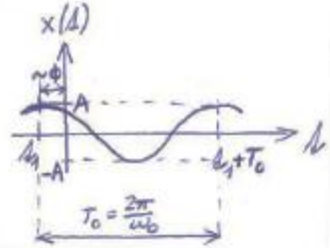
- Sinusoids
- Phase shift and time shift
- Complex numbers, Euler’s formula
- Complex exponential signal
- Phasors, rotating phasors
- Inverse Euler formulas
- Rotating phasors interpretation of sinusoids
- Spectrum representation of sinusoids
- Two-sided spectrum
- One-sided spectrum

SINUSOIDS

Sinusoids, sinusoidal signals: sine or cosine signals
- crucial importance in the theory of signals

General formula of cosine signal:

$x(t) = A\cos(\omega_0 t + \phi)$



← positive real number

A ... amplitude - scaling factor, how large cosine signal is

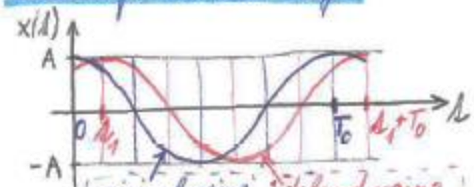
ω_0 ... radian frequency $\omega_0 = \frac{\Delta\varphi}{\Delta t} = \frac{2\pi}{T_0}$ ← fundamental period [s]

$\Delta\varphi$ [rad], Δt [s] ⇝ ω_0 [$\frac{rad}{s}$]

fundamental frequency $f_0 = \frac{1}{T_0}$ [Hz] , $\omega_0 = 2\pi f_0$

ϕ ... phase shift - phase (angle) at time instant $t = 0$ s

Phase shift and time shift



$t_1 = -\frac{\phi}{\omega_0}$

$\phi = -\omega_0 \cdot t_1 = -2\pi \frac{t_1}{T_0}$

original cosine: $A\cdot\cos(\omega_0 t)$; delayed cosine: $A\cdot\cos(\omega_0 t + \phi) = A\cdot\cos(\omega_0(t - t_1))$

phase shift, time shift

t_1 ... time shift ... time of zero phase (of cosine signal)

if $t_1 > 0$ ... signal delayed in time, shift the original signal to the right

if $t_1 < 0$ ... signal advanced in time, shift the original signal to the left

ϕ ... phase shift - instantaneous phase at time instant $t = 0$.

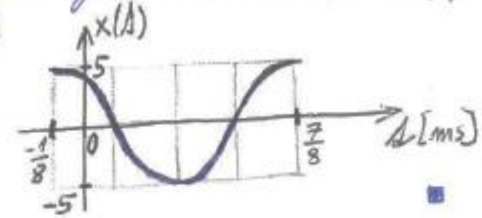
- ambiguous quantity, possible to add arbitrary integer multiple of 2π

→ principal value of the phase shift - satisfies $-\pi \le \phi \le \pi$

Ex. 2-1 Sketch the plot of continuous time signal $x(t) = 5\cos(2\pi\cdot 1000 t + \frac{\pi}{4})$ within duration of one period.

Sol.: $t_1 = \frac{-\phi}{\omega_0} = \frac{-\frac{\pi}{4}}{2\pi\cdot 1000} = -\frac{1}{8}$ ms

$T_0 = \frac{1}{f_0} = \frac{1}{1000} = 1$ ms



Ex. 2-2 Write the expression of sine function $A\cdot\sin(\omega_0 t + \phi)$ in terms of cosine function.

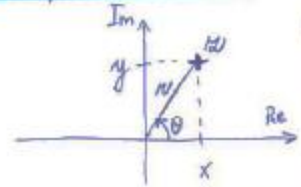
Sol.: In fact, sin is just cos shifted by angle $\frac{\pi}{2}$ to the right (delayed).

$A\cdot\sin(\omega_0 t + \phi) = A\cdot\cos(\omega_0 t + \phi - \frac{\pi}{2})$

Q: When $\phi = +\frac{\pi}{8}$, is the cosine signal delayed or advanced in time?

A: Signal is advanced in time ($t_1 < 0$, shifted to the left)

Complex numbers (note: imaginary unit $\sqrt{-1}$ is denoted as j)



One complex number z has two different notations:

1) Cartesian (Rectangular) form: $z = x + j\cdot y$

2) Polar (exponential) form: $z = r\cdot e^{j\theta}$

Conversions: Pol. → Rec.: $x = r\cdot\cos\theta$

$y = r\cdot\sin\theta$

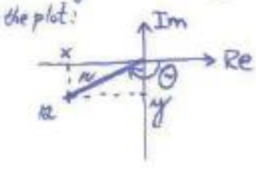
apparent from the picture above

Rec. → Pol.: $r = \sqrt{x^2 + y^2}$

$\Theta = \operatorname{arctg}\frac{y}{x}$

this is valid only for the range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

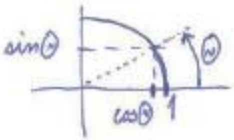
See the plot:



02-1

Euler's formula

$e^{j\Theta} = \cos\Theta + j\cdot\sin\Theta$ ... Euler's formula – again converting complex number from Cartesian to polar form, now without any limitation for Θ.

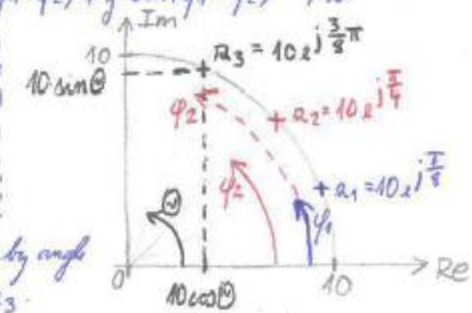


Ex. 2-3 Sketch $z_3 = r\cdot e^{j\Theta}$ in complex plane, substitute $r = 10$, $\Theta = \varphi_1 + \varphi_2$, $\varphi_1 = \frac{\pi}{8}$, $\varphi_2 = \frac{\pi}{4}$. Label the plot properly.

Sol.: $e^{j(\varphi_1+\varphi_2)} = \cos(\varphi_1+\varphi_2) + j\cdot\sin(\varphi_1+\varphi_2) \quad /\cdot r$

Note:
$z_3 = 10\cdot e^{j\frac{3}{8}\pi} =$
$= 10\cdot e^{j\frac{\pi}{8}}\cdot e^{j\frac{\pi}{4}} =$
$= z_1\cdot e^{j\frac{\pi}{4}}$

↳ take z_1, rotate it by angle of $\frac{\pi}{4}$, you will obtain z_3.

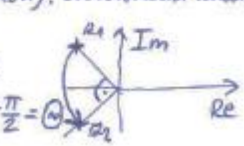


We see from Ex. 2-3, what means multiplying by complex exponential $e^{j\Theta}$, which is very important!!!

Multiplying $\cdot e^{j\Theta}$ is the following operation:

- input: some (complex) number, e.g. z_1 in Ex 2-3
- output: another complex number achievable as follows: Make circle passing through input number, mark oriented angle Θ ($\Theta > 0$ ... counter-clockwise, $\Theta < 0$ ... clockwise direction). Arrowhead indicates the output number.

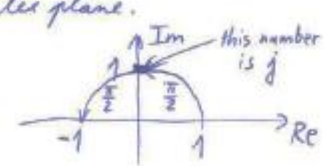
Example:



$z_2 = z_1\cdot e^{j\frac{\pi}{2}}$

Q.: Explain why $j^2 = -1$. Use sketch in complex plane.

A.: $j^2 = j\cdot j = e^{j\frac{\pi}{2}}\cdot e^{j\frac{\pi}{2}} = e^{j\left(\frac{\pi}{2}+\frac{\pi}{2}\right)} = e^{j\pi} = 1\cdot e^{j\pi}$



Complex exponential signal

$z(t) = A\cdot e^{j\cdot(\omega_0 t + \phi)}$ ..... complex exponential signal

$z(t) = A\cdot\cos(\omega_0 t + \phi) + j\cdot A\cdot\sin(\omega_0 t + \phi)$

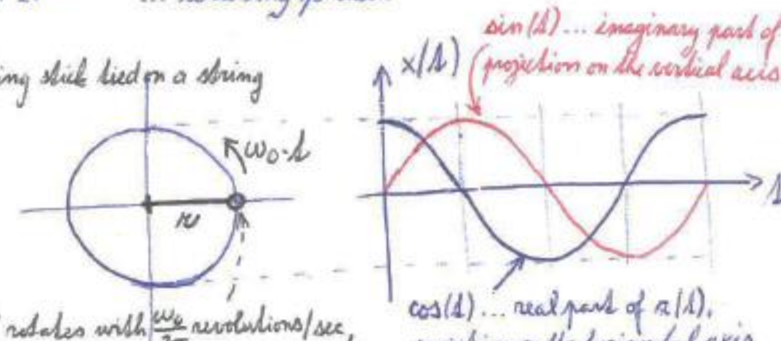
We see, that $x(t) = A\cdot\cos(\omega_0 t + \phi) = Re\{z(t)\}$

Phasors, rotating phasors

$X = A\cdot e^{j\cdot\phi}$ .... phasor, or complex amplitude

Then $z(t) = X\cdot e^{j\omega_0 t}$ ... rotating phasor

Demo: Rotating stick tied on a string for $\phi = 0$



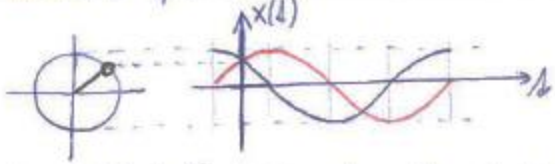
point rotates with $\frac{\omega_0}{2\pi}$ revolutions/sec, the position of the point can be seen in complex plane and expressed as $z(t) = r\cdot e^{j(\omega_0 t + \phi)}$

here in this demo $\phi = 0$

02-2

Q: Consider $\phi = \frac{\pi}{4}$ in above demonstration. What will change?

A:



Complex amplitude (initial point position for $t=0$) will rotate by $+\frac{\pi}{4}$. Furthermore, vertical axis in right part of the picture will move by $\frac{\frac{\pi}{4}}{2\pi}\cdot T = \frac{T}{8}$ to the right.

Ex. 2-4 Addition of two sinusoids via phasors

A signal $x(t)$ is defined as $x(t) = 2\cos(\omega_0 t + \frac{\pi}{3}) + \sqrt{2}\cos(\omega_0 t - \frac{3}{4}\pi)$

a) Use phasors to express $x(t)$ in the form $x(t) = A\cdot\cos(\omega_0 t + \phi)$.

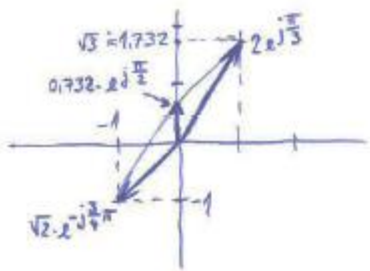
Sol.: $x(t) = Re\{e^{j\omega_0 t}\cdot(2\cdot e^{j\frac{\pi}{3}} + \sqrt{2}\cdot e^{-j\frac{3}{4}\pi})\} =$

$\doteq Re\{e^{j\omega_0 t}\cdot(1 + 1{,}732j - 1 - j)\} = Re\{e^{j\omega_0 t}\cdot 0{,}732\cdot e^{j\frac{\pi}{2}}\} =$

$= 0{,}732\cdot\cos(\omega_0 t + \frac{\pi}{2})$ ■

b) Plot all the phasors used to solve the problem a) in the complex plane

Sol.:



■

c) Find a complex-valued signal $z(t)$ such that $x(t) = Re\{z(t)\}$

Sol.: Found already in a) ... $z(t) = 0{,}732\cdot e^{j(\omega_0 t + \frac{\pi}{2})}$ ■

Inverse Euler formulas

– the way how to express sinusoids in terms of complex exponential signals.

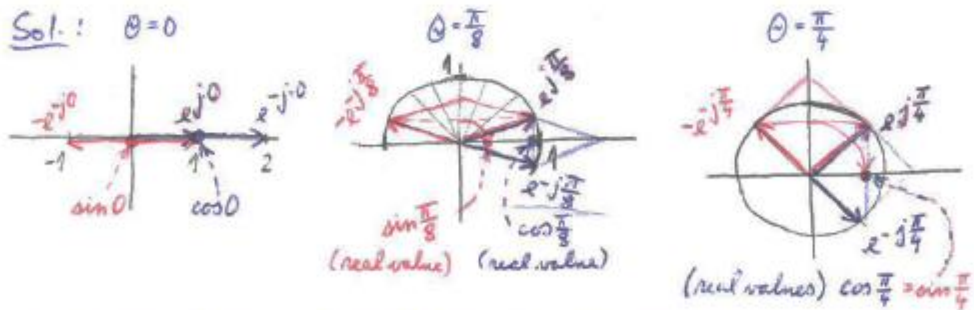
$$\cos\Theta = \frac{e^{j\Theta} + e^{-j\Theta}}{2}$$

$$\sin\Theta = \frac{e^{j\Theta} - e^{-j\Theta}}{2j} \quad \cdot\frac{1}{j} = -j \quad \frac{e^{j\Theta} - e^{-j\Theta}}{2} = e^{-j\frac{\pi}{2}}\cdot\frac{e^{j\Theta} - e^{-j\Theta}}{2}$$

} Inverse Euler formulas

Ex. 2-5: Show inverse Euler formulas in a graphical way for $\Theta = 0$, $\Theta = \frac{\pi}{8}$ and $\Theta = \frac{\pi}{4}$.

Sol.:



Rotating phasors interpretation of sinusoid

Using inverse Euler formula to express $A\cos(\omega_0 t + \phi)$ we obtain:

$$A\cos(\omega_0 t + \phi) = A\cdot\frac{e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}}{2} = \frac{A}{2}\left(X\cdot e^{j\omega_0 t} + X^{*}\cdot e^{-j\omega_0 t}\right)$$

complex conjugate

note: • the first complex exponential rotates in positive direction, counter-clockwise
• the second complex exponential rotates in negative direction, clockwise

Spectrum representation of sinusoid

Spectrum – is a compact representation of the frequency content of a signal that is composed of sinusoids.

– simply the collection of amplitude, frequency and phase information, that allows us to express the signal in the form $x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k) = X_0 + Re\left\{\sum_{k=1}^{N} X_k e^{j2\pi f_k t}\right\} =$

$$= \boxed{X_0 + \sum_{k=1}^{N}\left(\frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t}\right)}$$

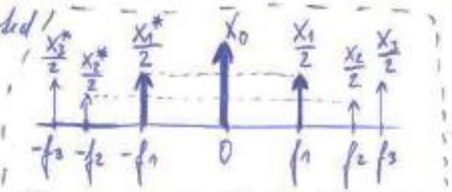
average value, „DC" component

Two-sided spectrum

– signal representation in the form of dependency of complex amplitudes on frequencies

– just set of pairs: $\boxed{\left\{ (0, X_0), \left(f_1, \frac{X_1}{2}\right), \left(-f_1, \frac{X_1^*}{2}\right), \dots \right\}}$

– can be ploted

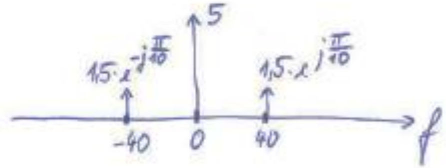


different representations of the spectrum

Ex. 2-6: Plot two-sided spectrum of the signal $x(t) = 5 + 3\cos\left(80\pi t + \frac{\pi}{10}\right)$

Sol.: $x(t) = 5 + 3 \cdot \frac{e^{j\left(80\pi t + \frac{\pi}{10}\right)} + e^{-j\left(80\pi t + \frac{\pi}{10}\right)}}{2} = 5 + 1{,}5\, e^{j\left(80\pi t + \frac{\pi}{10}\right)} + 1{,}5\, e^{j\left(-80\pi t - \frac{\pi}{10}\right)} =$

$= 5 + 1{,}5\, e^{j\left(2\pi \cdot 40 t + \frac{\pi}{10}\right)} + 1{,}5\, e^{j\left(-2\pi \cdot 40 t - \frac{\pi}{10}\right)} = 5 + 1{,}5 \cdot e^{j\frac{\pi}{10}} \cdot e^{j2\pi 40 t} + 1{,}5\, e^{-j\frac{\pi}{10}} \cdot e^{-j2\pi 40 t}$



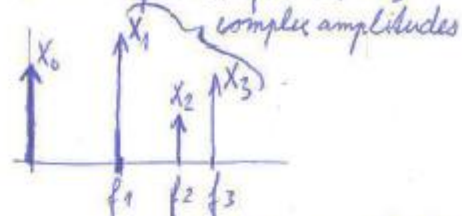
note: „DC" component is in full size, other are halved

One-sided spectrum

– recall the form $x(t) = X_0 + Re\left\{\sum_{k=1}^{N} X_k \cdot e^{j2\pi f_k t}\right\}$

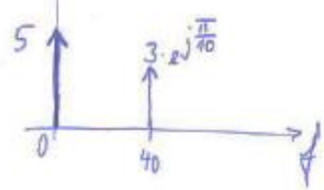
→ just set of pairs: $\{(0, X_0), (f_1, X_1), (f_2, X_2), \dots\}$

– can be ploted



Ex. 2-7 Plot one-sided spectrum of the signal $x(t) = 5 + 3\cos\left(80\pi t + \frac{\pi}{10}\right)$

Sol.: $x(t) = 5 + Re\left\{3 \cdot e^{j\left(2\pi \cdot 40 t + \frac{\pi}{10}\right)}\right\} = 5 + Re\left\{3 \cdot e^{j\frac{\pi}{10}} \cdot e^{j \cdot 2\pi 40 t}\right\}$



note: every components are in full size

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Vocabulary EN/CZ

sinusoid	sinusoida
amplitude	amplituda
radian frequency	úhlová frekvence
(principal) phase shift	(základní) počáteční fáze
time shift	časový posun (čas nulové fáze)
delayed (signal)	zpožděný (signál)
advanced (signal)	předbíhající (signál)
Cartesian form	Kartézský (součtový) tvar komplexního čísla
polar form	Polární (exponenciální) tvar komplexního čísla
counter-clockwise	proti směru hodinových ručiček
clockwise	ve směru hodinových ručiček
phasor	fázor
complex conjugate	komplexně sdružené číslo
two-sided spectrum	dvoustranné spektrum

References

- McClellan, Schafer and Yoder, Signal Processing First, ISBN 0-13-065562-7., Prentice Hall, Upper Saddle River, NJ 07458. 2003 Pearson Education, Inc.