Sampling and Aliasing

Signals and codes (SK)

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Lecture 4

Lecture goal and content

Goal

- Be able to set up correct sample frequency for voltage measurement using AD converters and choose sufficient resolution of AD converter.
- Understand Shannon sampling theorem

Content

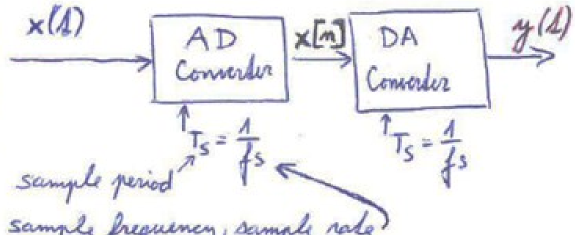
- Sampling and Reconstruction
- Sampling sinusoidal signals
- Spot on a disk
- Aliasing and Folding Concept
- Spectrum of Discrete-time signal
- Shannon Sampling Theorem
- Quantization

SAMPLING AND ALIASING

Sampling and reconstruction

Basic idea →	Music played	Recording music	DATA	Playing music	Music heard
Transport examples →	Some signal	Data acquisition		Signal reconstruction	Signal reconstructed

- Transmitter/receiver voltage/current
- Railway traction current
- Battery voltage response
- Anything else



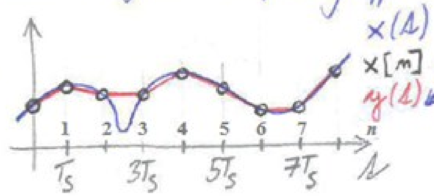
AD ... analog to digital (continuous to discrete)

DA ... digital to analog (discrete to continuous)

Q.: Is $y(t)$ really the same as $x(t)$?

A.: $y(t)$ and $x(t)$ can be the same, or they can be almost the same or they can be completely different. See lecture below.

Demo:



$y(t)$ depends on interpolation method, here linear interpolation is shown.

- discrete time signal $x[n]$ shall correspond to continuous time signal $x(t)$. (data acquisition)
- DAQ device acquires data with sample period $T_S = \frac{1}{f_S}$ ← (mostly stated in specification)

⇝ $x[n] = x(n \cdot T_S)$ .... this applies for integer n.

Sampling sinusoidal signals $x(t) = A\cos(\omega t + \phi)$

$x[n] = x(nT_S) = A \cdot \cos(\omega n T_S + \phi) = A\cos(\hat{\omega} n + \phi)$, where we define Normalized radian frequency $\hat{\omega}$ (also discrete time frequency) as

$$\hat{\omega} = \omega \cdot T_S = \frac{\omega}{f_S} = \frac{2\pi f_0}{f_S} = \frac{2\pi T_S}{T_0}$$ ... dimensionless (unit is radian)

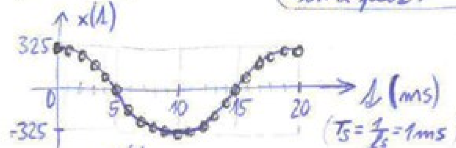
↳ meaning of $\hat{\omega}$ is phase step of adjacent samples.

Ex. 4_1: We sample $x(t) = 325\cos(2\pi 50 t)$ with different f_S. Find $\hat{\omega}$, write expression for discrete time signal $x[n]$. How many samples per period? Sketch samples of $x(t)$ in a plot.

a) $f_S = 1\,kHz$. Sol.: $\hat{\omega} = \frac{2\pi f_0}{f_S} = \frac{2\pi \cdot 50}{1000} = 0{,}1\pi$

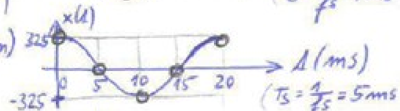
$x[n] = 325\cos(0{,}1\pi n)$

$\frac{T_0}{T_S} = \frac{f_S}{f_0} = \frac{1000}{50} = 20$ samples/period



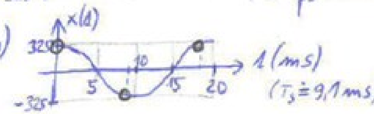
b) $f_S = 200\,Hz$. Sol.: $\hat{\omega} = \frac{2\pi \cdot 50}{200} = \frac{\pi}{2}$, $x[n] = 325\cos(0{,}5\pi n)$

$\frac{200}{50} = 4$ samples/period



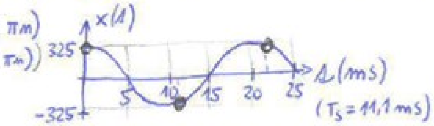
c) $f_S = 111{,}11\,Hz$. Sol.: $\hat{\omega} = \frac{2\pi \cdot 50}{111{,}11} = 0{,}9\pi$, $x[n] = 325\cos(0{,}9\pi n)$

$\frac{111{,}11}{50} = 2{,}22$ samples/period



d) $f_S = 90{,}91\,Hz$. Sol.: $\hat{\omega} = \frac{2\pi \cdot 50}{90{,}91} = 1{,}1\pi$, $x[n] = 325\cos(1{,}1\pi n)$ $(= 325\cos(0{,}9\pi n))$

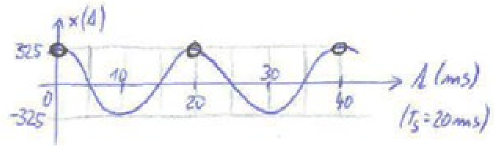
$\frac{90{,}91}{50} = 1{,}8$ samples/period



e) $f_S = 50\,Hz$. Sol.: $\hat{\omega} = \frac{2\pi \cdot 50}{50} = 2\pi$

$x[n] = 325 \cdot \cos(2\pi n)$ $(= 325)$

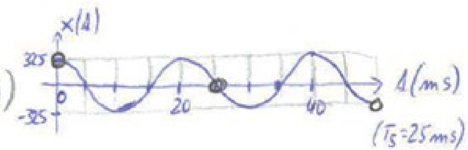
$\frac{50}{50} = 1$ sample/period



f) $f_S = 40\,Hz$. Sol.: $\hat{\omega} = \frac{2\pi \cdot 50}{40} = 2{,}5\pi$

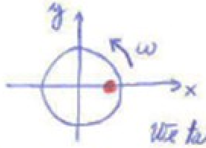
$x[n] = 325\cos(2{,}5\pi n)$ $(= 325\cos(0{,}5\pi n))$

$\frac{40}{50} = 0{,}8$ samples/period



04_1

Spot on a disk – introduction to sampling, reconstruction, aliasing and folding



Imagine a disk with a red spot rotating counter-clockwise with $\omega = 2\pi f_0$. Let $f_0 = 1\,Hz$.
We take photos with a camera and then try to reconstruct the movement.

1.) Example of correct sampling $\quad f_s = 4\,Hz \rightsquigarrow T_s = \frac{1}{4}\,s$, $\hat{\omega} = 2\pi \cdot \frac{f_0}{f_s} = 2\pi \cdot \frac{1}{4} = 0{,}5\pi$

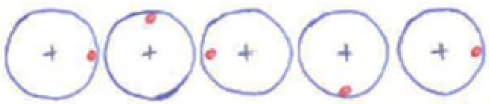
note: we will see later on, that for $f_s > 2f_0$ will be (obtained $0 \le \hat{\omega} < \pi$ and the signal will be correctly reconstructed.)

$t = 0\,s$, $m = 0$ $\quad t = \frac{1}{4}\,s$, $m = 1$ $\quad t = \frac{2}{4}\,s$, $m = 2$ $\quad t = \frac{3}{4}\,s$, $m = 3$ $\quad t = 1\,s = T_0$, $m = 4$

Reconstruction:
$T_s = \frac{1}{4}\,s$
$T_0 = 4 \cdot \frac{1}{4} = 1\,s$ … correct
4 samples/period

2.) Example of aliasing $\quad f_s = 0{,}8\,Hz \rightsquigarrow T_s = 1{,}25\,s$, $\hat{\omega} = \frac{2\pi \cdot 1}{0{,}8} = 2{,}5\pi$

(note: $f_s < f_0$, $2\pi < \hat{\omega} < 3\pi$)



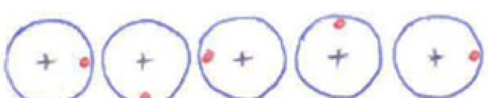
$t = 0\,s$, $m = 0$ $\quad t = 1{,}25\,s$, $m = 1$ $\quad t = 2{,}5\,s$, $m = 2$ $\quad t = 3{,}75\,s$, $m = 3$ $\quad t = 5\,s = T_0$, $m = 4$

Reconstruction:
$T_s = 1{,}25\,s$
$T_0 = 4 \cdot 1{,}25 = 5\,s$ … wrong, originally the period was 1s.

It seems that the disk rotates in correct direction, but with much lower speed.

3.) Example of folding $\quad f_s = \frac{4}{3}\,Hz \rightsquigarrow T_s = 0{,}75\,s$, $\hat{\omega} = 2\pi \cdot \frac{1}{\frac{4}{3}} = 1{,}5\pi$

(note: $f_0 < f_s < 2f_0$, $\pi < \hat{\omega} < 2\pi$)



$t = 0\,s$, $m = 0$ $\quad t = 0{,}75\,s$, $m = 1$ $\quad t = 1{,}5\,s$, $m = 2$ $\quad t = 2{,}25\,s$, $m = 3$ $\quad t = 3\,s = T_0$, $m = 4$

Reconstruction:
$T_s = 0{,}75\,s$
$T_0 = 4 \cdot 0{,}75 = 3\,s$ … wrong, originally the period was 1s

It seems that the disk rotates in opposite direction with lower speed.

Aliasing and folding concept

Reason for aliasing and folding → sinusoids are periodic with 2π and symetric.

Aliasing: $\cos(\theta + 2\pi) = \cos\theta$ applies for all θ

Recall Ex. 4.1 f) $x[n] = 325\cos(2{,}5\pi n) = 325\cos(0{,}5\pi n + 2\pi n) = 325\cos(0{,}5\pi n)$ ← which is the same as in Ex. 4.1 b)

Thus for the same values $x[n]$ we can have signals: $\cos(0{,}5\pi n)$ … principal alias, $0 < \hat{\omega} < \pi$
$\cos(2{,}5\pi n)$ … 1st alias
$\cos(4{,}5\pi n)$ … 2nd alias
⋮
$\cos(\ell\pi n)$ … ℓth alias

Aliasing as undesirable phenomenon occurs if $2k\pi < \hat{\omega} < (2k+1)\pi$ for nonzero integer k

Folding: $\cos(2\pi - \theta) = \cos\theta$ applies for all θ

Recall Ex. 4.1 d) $x[n] = 325\cos(1{,}1\pi n) = 325\cos(2\pi n - 1{,}1\pi n) = 325\cos(0{,}9\pi n)$ ← which is the same as in Ex. 4.1 c)

Thus for the same values $x[n]$ we can have signals: $\cos(0{,}9\pi n)$ … principal alias, $0 < \hat{\omega} < \pi$
$\cos(1{,}1\pi n) = \cos(-0{,}9\pi n + 2\pi n)$ … 1st folded alias
$\cos(3{,}1\pi n) = \cos(-0{,}9\pi n + 4\pi n)$ … 2nd folded alias
⋮
$\cos(1{,}1\pi n + (\ell-1)2\pi n) = \cos(-0{,}9\pi n + \ell \cdot 2\pi n)$ … ℓth folded alias

Relation between aliasing and folding – applies $\hat{\omega}_{(k^{th}\,alias)} = -\hat{\omega}_{(-k^{th}\,folded\,alias)}$

Demo: • again known example of $x[n] = \cos 0{,}5\pi n \rightsquigarrow \hat{\omega} = 0{,}5\pi$ … principal alias

1st alias: $\hat{\omega}_{1a} = 0{,}5\pi + 2\pi = 2{,}5\pi$ $\qquad$ (-1)st alias: $\hat{\omega}_{-1a} = 0{,}5\pi - 2\pi = -1{,}5\pi$ $\qquad$ compare
1st folded alias: $\hat{\omega}_{1f} = -0{,}5\pi + 2\pi = 1{,}5\pi$ $\qquad$ (-1)st folded alias: $\hat{\omega}_{-1f} = -0{,}5\pi - 2\pi = -2{,}5\pi$

• $x[n] = \cos 0{,}9\pi n$ … $\hat{\omega} = 0{,}9\pi$ … $\hat{\omega}_{1a} = 2{,}9\pi$ $\quad \hat{\omega}_{-1a} = -1{,}1\pi$
$\hat{\omega}_{1f} = 1{,}1\pi$ $\quad \hat{\omega}_{-1f} = -2{,}9\pi$

04_2

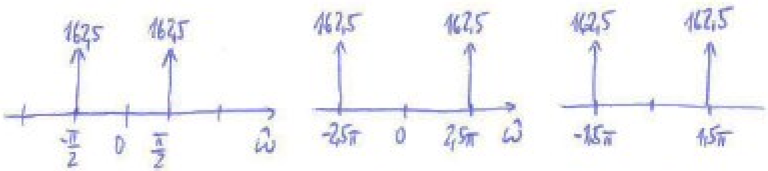
Spectrum of a discrete-time signal

Consider again $x[n] = 325\cos(0{,}5\pi n)$. What is the spectrum?

$325\cos 0{,}5\pi n = \frac{1}{2}\left(325 e^{j0{,}5\pi n} + 325 e^{-j0{,}5\pi n}\right) =$ (principal alias)

can be also $= \frac{1}{2}\left(325 e^{j2{,}5\pi n} + 325 e^{-j2{,}5\pi n}\right) =$ (1^{st} alias)

can be also $= \frac{1}{2}\left(325 e^{j1{,}5\pi n} + 325 e^{-j1{,}5\pi n}\right)$ (1^{st} folded alias)



Spectrum of principal alias — 1^{st} alias — 1^{st} folded alias

We have to choose one alias, one spectrum from many possible ones.

We choose always principal alias (and its spectrum), because it is the alias, that is consequently used for reconstruction of the discrete-time signal.

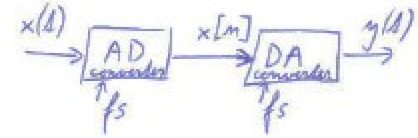
Shannon Sampling Theorem

A continuous-time signal with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = \frac{1}{T_s}$, that is greater than $2f_{max}$.

note: $2f_{max}$ is called Nyquist rate

note: in practice higher sampling is recommended for better accuracy, especially if we are not sure that the signal is composed of pure sinusoids

Ex. 4_2



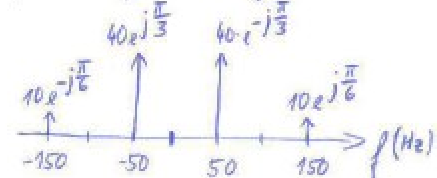
Consider $x(t) = 80\cos\left(2\pi 50 t - \frac{\pi}{3}\right) + 20\cos\left(2\pi 150 t + \frac{\pi}{6}\right)$.

Plot spectrums of $x(t)$, $x[n]$ and $y(t)$ and find $y(t)$ for different rates of converters f_s

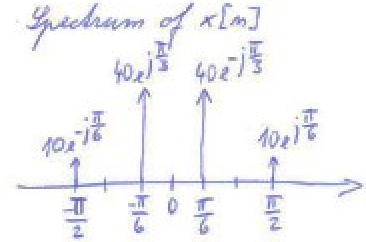
a) $f_s = 600\ Hz$

Sol.: For 50 Hz: $\hat{\omega} = 2\pi\cdot\frac{50}{600} = \frac{\pi}{6}$
For 150 Hz: $\hat{\omega} = 2\pi\cdot\frac{150}{600} = \frac{\pi}{2}$
} both inside $\langle 0;\pi\rangle$ ⟶ both principal alias ⟶ $y(t) = x(t)$

Spectrum of $x(t)$ and $y(t)$:



Spectrum of $x[n]$



b) $f_s = 250\ Hz$

Sol.: Spectrum of $x(t)$ is the same as in subtask a).

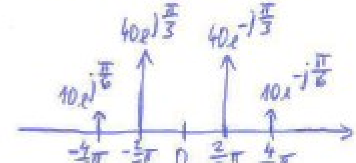
For 50 Hz: $\hat{\omega} = 2\pi\cdot\frac{50}{250} = \frac{2}{5}\pi$ ... inside $\langle 0;\pi\rangle$ ⟶ principal alias

For 150 Hz: $\hat{\omega} = 2\pi\cdot\frac{150}{250} = \frac{6}{5}\pi$ ... lies outside $\langle 0;\pi\rangle$!

⟶ $-\frac{6}{5}\pi + 2\pi = \frac{4}{5}\pi$ is principal alias for 150 Hz ($\frac{6}{5}\pi$ is 1^{st} folded alias)

$\cos\left(\frac{6}{5}\pi n + \frac{\pi}{6}\right) = \cos\left(2\pi n - \left(\frac{6}{5}\pi n + \frac{\pi}{6}\right)\right) = \cos\left(\frac{4}{5}\pi n - \frac{\pi}{6}\right)$ folding is changing sign!

Spectrum of $x[n]$:



04_3

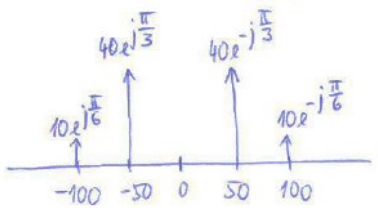
Reconstructing $y(t)$ from $x[n]$ with $f_s = 250\,Hz$:

$\hat{\omega} = 2\pi \frac{f_0}{f_s} \rightsquigarrow f_0 = \frac{\hat{\omega} \cdot f_s}{2\pi}$

For 50 Hz: $f_0 = \frac{\frac{2}{5}\pi \cdot 250}{2\pi} = 50\,Hz$ ... O.K.

For 150 Hz: $f_0 = \frac{\frac{4}{5}\pi \cdot 250}{2\pi} = 100\,Hz$ ... wrong (it was originally 150 Hz)

Spectrum of $y(t)$:



Expression for $y(t)$: $y(t) = \underbrace{80\cos(2\pi 50 t - \frac{\pi}{3})}_{\text{correct}} + \underbrace{20\cos(2\pi 100 t - \frac{\pi}{6})}_{\text{1st folded alias (folding)}}$ ■

Ex. 4_3 What is minimum sample rate necessary for correct reconstructing of the signal $x(t)$ from Ex. 4_2?

Sol.: Sample rate has to be at least twice higher than the maximum frequency component of the signal $x(t)$ according to the Shannon sampling theorem.

Thus: $f_s > 300\,Hz$ ■

Quantization

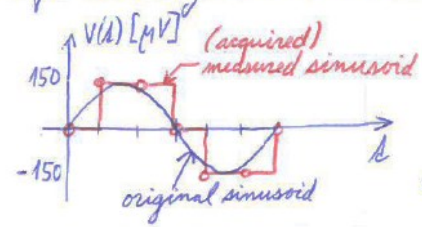
- the lecture above was focused on discretization of time.
- AD converter discretizes also values of the signal – quantization occurs.
  - a component of DAQ device built of set of comparators
- bit resolution is crucial parameter concerning quantization
- resolution (which least value can be detected by the device) of AD converter for the given measurement range is given by the formula

$$\text{resolution} = \frac{\text{measurement range}}{2^{\text{bit resolution}}}$$

Ex. 4_4 PCIe DAQ device NI6341 measures voltage at analog input with 16 bit resolution

a) What is resolution of the measured voltage for chosen measurement range from -5 to +5 V? Sketch the shape of measured sinusoidal voltage of amplitude 150 μV for $f_s = 6 \cdot f_0$ assuming there is no noise.

Sol.: $\text{resolution} = \frac{5-(-5)}{2^{16}} \doteq 0{,}15\,mV$



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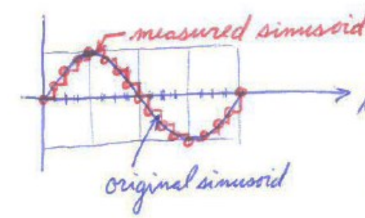
b) Repeat subtask a) for measurement range from -200 to +200 mV and for $f_s = 16 \cdot f_0$.

Sol.: $\text{resolution} = \frac{0{,}4}{2^{16}} = 6\,\mu V$



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Note:

Bit resolution	# discrete levels	example
8	256	some microcontroller
10	1024	arduino
16	65 536	middle precision DAQ
24	16 777 216	higher precision DAQ

04_4

Vocabulary EN/CZ

DAQ – Data acquisition	Pořizování dat
Normalized radian frequency	Normalizovaná úhlová frekvence
Integer	Celé číslo
Dimensionless	Bezrozměrný
Expression	Výraz
Aliasing	Aliasing (nepřekládá se)
Folding	Folding je druhem aliasingu
Sample rate	Vzorkovací frekvence
Quantization	Kvantování
Bit resolution	Bitové rozlišení
Resolution	Rozlišení (pro přístroj daná nejnižší hodnota změny měřené veličiny, která vyvolá změnu údaje přístroje)
Measurement range	Měřicí rozsah

References

- McClellan, Schafer and Yoder, Signal Processing First, ISBN 0-13-065562-7., Prentice Hall, Upper Saddle River, NJ 07458. 2003 Pearson Education, Inc.