

Spectrum of periodic signals

Signals and codes (SK)

Department of Transport Telematics
Faculty of Transportation Sciences, CTU in Prague

Lecture 3



Lecture goal and content

Goal

- Be able to find spectral representation of arbitrary periodical signal and reconstruct the signal back from the spectrum.

Content

- Multiplication of sinusoids – beat note
- Amplitude modulation (AM) principle
- Periodic and nonperiodic signals
- Fourier series
- Fourier analysis
- Fourier synthesis
- From Fourier series to Fourier transform
- From Fourier transform to Short time Fourier transform

SPECTRUM OF PERIODIC SIGNALS

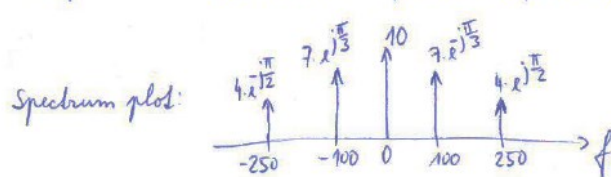
Ex.3-1 Find two-sided spectrum (recall the last lecture)

of $x(t) = 10 + 14 \cos(200\pi t - \frac{\pi}{3}) + 8 \cos(500\pi t + \frac{\pi}{2})$

Sol.: $x(t) = 10 + \frac{14}{2} (e^{j(200\pi t - \frac{\pi}{3})} + e^{-j(200\pi t - \frac{\pi}{3})}) + \frac{8}{2} (e^{j(500\pi t + \frac{\pi}{2})} + e^{-j(500\pi t + \frac{\pi}{2})}) =$

$= 10 + 7 \cdot e^{-j\frac{\pi}{3}} \cdot e^{j2\pi \cdot 100t} + 7 \cdot e^{j\frac{\pi}{3}} \cdot e^{-j2\pi \cdot 100t} + 4 \cdot e^{j\frac{\pi}{2}} \cdot e^{j2\pi \cdot 250t} + 4 \cdot e^{-j\frac{\pi}{2}} \cdot e^{-j2\pi \cdot 250t}$

and Spectrum is $\{(0, 10), (100, 7e^{-j\frac{\pi}{3}}), (-100, 7e^{j\frac{\pi}{3}}), (250, 4e^{j\frac{\pi}{2}}), (-250, 4e^{-j\frac{\pi}{2}})\}$



Multiplication of sinusoids - beat note

Consider $x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$

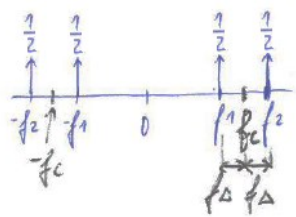
Denote $f_1 = f_c - f_\Delta$, $f_2 = f_c + f_\Delta$

center frequency
frequency deviation

Then $f_c = \frac{1}{2}(f_1 + f_2)$, $f_\Delta = \frac{1}{2}(f_2 - f_1)$

$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) = \text{Re}\{e^{j2\pi f_1 t}\} + \text{Re}\{e^{j2\pi f_2 t}\} =$
 $= \text{Re}\{e^{j2\pi(f_c - f_\Delta)t}\} + \text{Re}\{e^{j2\pi(f_c + f_\Delta)t}\} = \text{Re}\{e^{j2\pi f_c t} (e^{-j2\pi f_\Delta t} + e^{j2\pi f_\Delta t})\} =$
 $= \text{Re}\{e^{j2\pi f_c t} \cdot 2 \cos(2\pi f_\Delta t)\} = 2 \cos(2\pi f_c t) \cos(2\pi f_\Delta t)$

multiplication leads to addition and vice versa



Ex.3-2: Find spectrum of a product $x(t) = \cos \pi t \cdot \sin 10\pi t$

Sol.: $x(t) = \cos \pi t \cdot \sin 10\pi t = \frac{e^{j\pi t} + e^{-j\pi t}}{2} \cdot \frac{e^{j10\pi t} - e^{-j10\pi t}}{2j} =$
 $= \frac{e^{j\pi t} + e^{-j\pi t}}{2} \cdot \frac{e^{j10\pi t} \cdot e^{-j\frac{\pi}{2}} + e^{-j10\pi t} \cdot e^{j\frac{\pi}{2}}}{2j} =$
 $= \frac{1}{4} (e^{j11\pi t} \cdot e^{-j\frac{\pi}{2}} + e^{j9\pi t} \cdot e^{j\frac{\pi}{2}} + e^{-j9\pi t} \cdot e^{-j\frac{\pi}{2}} + e^{-j11\pi t} \cdot e^{j\frac{\pi}{2}})$

Not necessary for this Ex.
 $= \frac{1}{2} \cos(11\pi t - \frac{\pi}{2}) + \frac{1}{2} \cos(9\pi t - \frac{\pi}{2}) =$
 $= \frac{1}{2} (\sin 11\pi t + \sin 9\pi t)$

note: original frequencies 0.5 Hz and 5 Hz are missing in the spectrum, but there is 5 ± 0.5 Hz components instead.
 f_c coming from $\sin 10\pi t$ f_Δ coming from $\cos \pi t$

Amplitude modulation (AM) principle

$v(t)$... transmitted signal (e.g. voice)

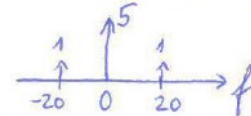
$\cos 2\pi f_c t$... carrier signal, f_c ... carrier frequency

AM signal: $x(t) = v(t) \cos(2\pi f_c t)$

nearby the same as the beat note, but including f_c frequency

Ex.3-3: Derive the spectrum of $v(t) = 5 + 2 \cos 2\pi 20t$ and plot it.

Sol.: $v(t) = 5 + e^{j2\pi 20t} + e^{-j2\pi 20t}$



Ex.3-4: Now take $v(t)$ from Ex.3-3 and derive the spectrum for $x(t) = v(t) \cdot \cos(2\pi 200t)$ and plot it.

Sol.: $x(t) = (5 + e^{j2\pi 20t} + e^{-j2\pi 20t}) \cdot \frac{1}{2} (e^{j2\pi 200t} + e^{-j2\pi 200t}) =$
 $= \frac{1}{2} (5e^{j2\pi 200t} + 5e^{-j2\pi 200t} + e^{j2\pi 220t} + e^{-j2\pi 220t} + e^{j2\pi 180t} + e^{-j2\pi 180t})$

note: compare spectrums in last two exercises

note: Real AM broadcasting --- $150 \text{ kHz} < f_c < 26 \text{ MHz}$, channel spacing 9 kHz
(base band) (broadcast)

Periodic and nonperiodic signals

• Periodic signals $x(t) = x(t + T_0)$, T_0 ... fundamental period
Such signals can be reconstructed from cosines of harmonically related frequencies, i.e. if all frequencies are integer multiple of f_0 , signal can be reconstructed using $N+1$ waveforms:

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi k f_0 t + \varphi_k), \text{ where}$$

f_0 ... fundamental frequency, greatest common divisor (g.c.d.) of all the signal frequencies

f_k ... k^{th} harmonic frequency

Ex.3-5: Signal contains frequencies $\{1, 2; 2; 6\}$ kHz. Find the fundamental frequency, state which harmonic frequencies does the signal contain. Is the signal periodic?

Sol.: $\text{g.c.d.}(1, 2; 2; 6) = 0,4 \text{ kHz} = f_0$

$\frac{1,2}{0,4} = 3^{\text{th}}$ harmonic $\frac{2}{0,4} = 5^{\text{th}}$ harmonic $\frac{6}{0,4} = 15^{\text{th}}$ harmonic. Signal is periodic

• Nonperiodic signals

What if one frequency in a signal exists, which is not a rational multiple of each other?

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi f_k t + \varphi_k) \text{ ... still valid, BUT no assumptions about } f_k$$

Periodicity is tied to harmonic frequencies!

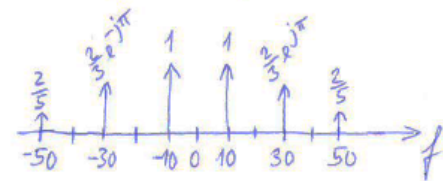
Ex.3-6: Consider two signals which seem nearly the same:

$$x_1(t) = 2 \cos(20\pi t) - \frac{2}{3} \cos(20\pi \cdot 3 \cdot t) + \frac{2}{5} \cos(20\pi \cdot 5 \cdot t)$$

$$x_2(t) = 2 \cos(20\pi t) - \frac{2}{3} \cos(20\pi \cdot \sqrt{8} \cdot t) + \frac{2}{5} \cos(20\pi \cdot 5 \cdot t)$$

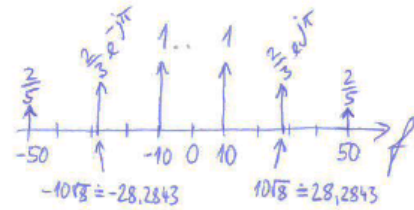
Sketch the spectrum plot of these signals, plot them in duration of 10 periods using SW of your choice. Observe, which one is periodic.

Sol.: Spectrum of $x_1(t)$

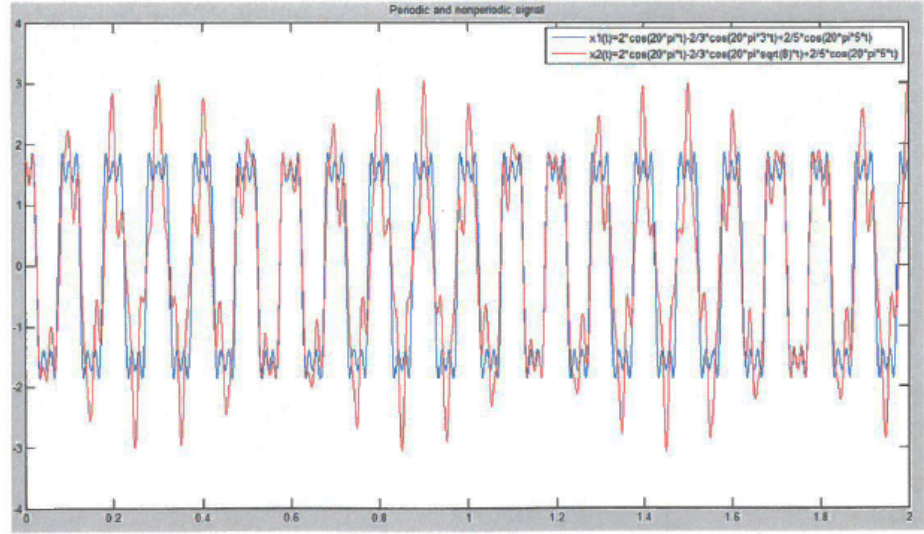


$f_0 = 10 \text{ Hz}$ and has to be periodic

Spectrum of $x_2(t)$



$f_0 \rightarrow 0$... can not be periodic



Fourier series

- above examples show, that we can synthesize PERIODIC signals by summing harmonically related sinusoids.

Fourier series build general theory, how any periodic signal can be synthesized using sum of harmonically related sinusoids.

Fourier series: $x(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{j2\pi f_0 k t}$, where

$f_0 \dots$ fundamental frequency, $f_0 = \frac{1}{T_0} \leftarrow$ fundamental period
 $f_k = k \cdot f_0 \dots k^{\text{th}}$ harmonic frequency

2 aspects of Fourier theory:

- Fourier analysis ... starting from $x(t)$... calculating $\{a_k\}$ set of
- Fourier synthesis ... starting from $\{a_k\}$... calculating $x(t)$

Q: What is the relation between $\{a_k\}$ and spectrum of the signal $x(t)$?

A: $\{a_k\}$ is set of complex amplitudes a_k . Each a_k is exactly a value of spectral line at respective frequency f_k !

Fourier analysis

How to calculate complex amplitudes a_k ?

By Fourier integral $a_k = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi f_0 k t} dt$

note: integral limits are arbitrary, when covering one whole period T_0

note: for $k=0$ we obtain $a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$, which is already known as an average value (or DC value) of periodic signal.

Fourier synthesis

How to calculate $x(t)$ from set of $\{a_k\}$?

We use original Fourier series formula $x(t) = \sum_{k=-\infty}^{+\infty} a_k \cdot e^{j2\pi f_0 k t}$

Then we use inverse Euler formulas and due to symmetry $a_k = a_{-k}^*$ we finally obtain some combination of cosine functions.

Ex. 3-7: Perform Fourier analysis of a signal $x(t) = \cos(2\pi f_0 t + \frac{\pi}{8})$

$$\begin{aligned} \text{Sol: } a_k &= \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi f_0 k t} dt = \frac{1}{T_0} \int_0^{T_0} \cos(2\pi f_0 t + \frac{\pi}{8}) \cdot e^{-j2\pi f_0 k t} dt = \\ &= \frac{1}{T_0} \int_0^{T_0} \frac{1}{2} (e^{j(2\pi f_0 t + \frac{\pi}{8})} + e^{-j(2\pi f_0 t + \frac{\pi}{8})}) \cdot e^{-j2\pi f_0 k t} dt = \\ &= \frac{1}{2T_0} \int_0^{T_0} e^{j\frac{\pi}{8}} \cdot e^{j2\pi f_0 t(1-k)} + e^{-j\frac{\pi}{8}} \cdot e^{-j2\pi f_0 t(1+k)} dt \end{aligned}$$

note: k represents all integers

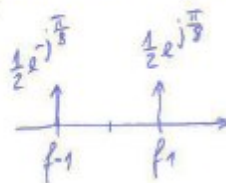
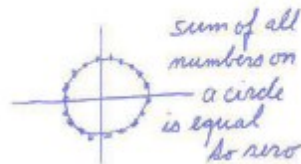
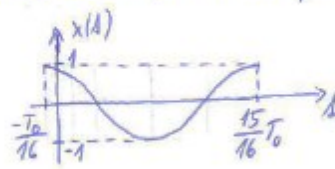
$$\text{For } k=1: a_1 = \frac{1}{2T_0} e^{j\frac{\pi}{8}} \int_0^{T_0} e^{j2\pi f_0 t(1-1)} dt = \frac{1}{2} e^{j\frac{\pi}{8}}$$

$$\text{For } k=-1: a_{-1} = \frac{1}{2} e^{-j\frac{\pi}{8}}$$

For all other k we obtain $a_k = 0$, because

$$\int_0^{T_0} e^{j2\pi f_0 t l} dt = 0 \text{ for all integers } l \neq 0.$$

Solution is done. We can plot original signal and the spectrum, i.e. $\{a_k\}$.

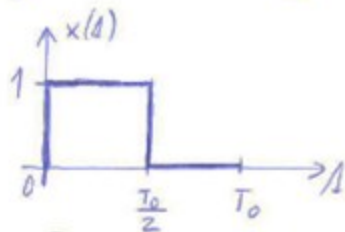


[03-3]

notes: • The spectrum is as expected, but we observed the procedure of computing Fourier integral.

• We observed so called orthogonality property of complex exponential, i.e. from the "long" expression inside the Fourier integral for the specific $k=l$ only the summand with $e^{j2\pi f_0 l (l-k)} = e^{j2\pi f_0 l \cdot 0} = e^{j \cdot 0} = 1$ leads to nonzero result of the integral.

Ex. 3-8: Find spectrum of graphically given continuous time signal $x(t)$ with a fundamental period T_0 .



Sol.: $a_k = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{-j2\pi f_0 k t} dt = \frac{1}{T_0} \int_0^{T_0/2} 1 \cdot e^{-j2\pi f_0 k t} dt =$

$$= \frac{1}{T_0} \cdot \frac{1}{-j2\pi f_0 k} \left[e^{-j2\pi f_0 k t} \right]_0^{T_0/2} = \frac{1}{-j2\pi k} (e^{-j2\pi f_0 k \frac{T_0}{2}} - 1) =$$

$$T_0 = \frac{1}{f_0}$$

$$= \frac{1}{-j2\pi k} (e^{-j\pi k} - 1)$$

for even $k \dots e^{-j\pi k} = 1 \Rightarrow a_k = 0$ (except for $k=0$)

for odd $k \dots e^{-j\pi k} = -1 \Rightarrow a_k = \frac{1}{-j2\pi k} (-1-1) =$

$$= \frac{1}{j\pi k} = \frac{1}{k\pi} \cdot e^{-j\frac{\pi}{2}}$$

So: $a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = 0.5$

$$a_1 = \frac{1}{\pi} e^{-j\frac{\pi}{2}}, a_{-1} = -\frac{1}{\pi} e^{-j\frac{\pi}{2}} = \frac{1}{\pi} e^{j\frac{\pi}{2}}$$

$$a_2 = 0, a_{-2} = 0$$

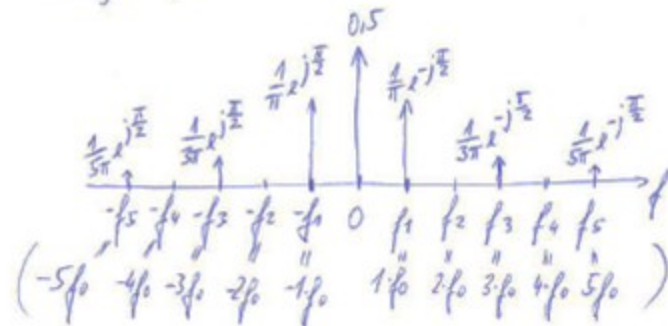
$$a_3 = \frac{1}{3\pi} e^{-j\frac{\pi}{2}}, a_{-3} = -\frac{1}{3\pi} e^{-j\frac{\pi}{2}}$$

$$a_4 = 0, a_{-4} = 0$$

$$a_5 = \frac{1}{5\pi} e^{-j\frac{\pi}{2}}, a_{-5} = -\frac{1}{5\pi} e^{-j\frac{\pi}{2}}$$

etc.

Plotting the first 5 harmonics we obtain

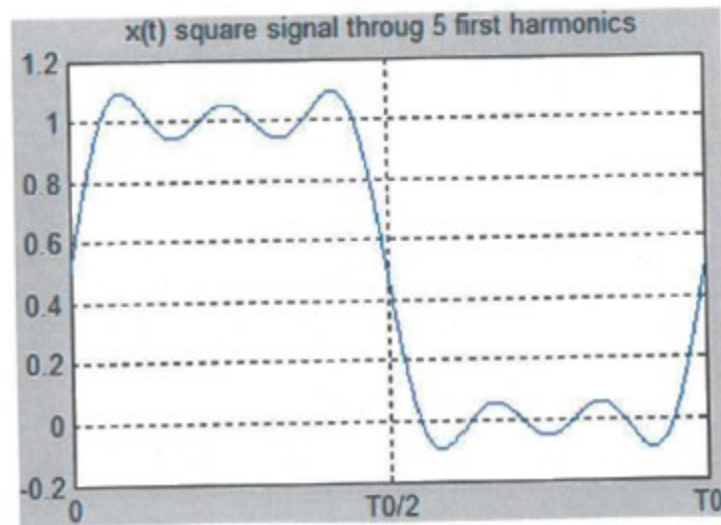


Ex. 3-9: Synthesize the signal from the resulting spectrum of the square signal (Ex. 3-8) through first 5 harmonics. Plot the result using SW of your choice.

Sol.: $x(t) = \sum_{k=-5}^5 a_k \cdot e^{j2\pi f_0 k t} = \frac{1}{5\pi} e^{j\frac{\pi}{2}} \cdot e^{j2\pi f_0 (-5)t} + \frac{1}{3\pi} e^{j\frac{\pi}{2}} \cdot e^{j2\pi f_0 (-3)t} +$

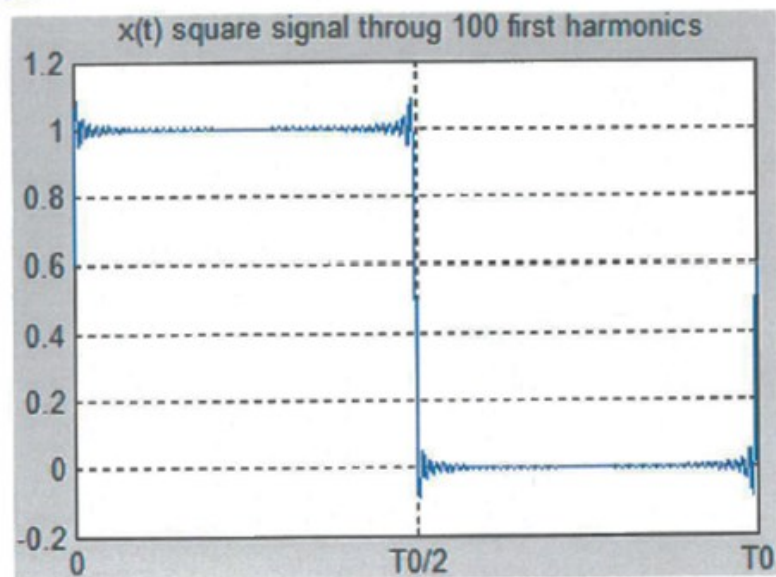
$$+ \frac{1}{\pi} e^{j\frac{\pi}{2}} \cdot e^{j2\pi f_0 (-1)t} + 0.5 + \frac{1}{\pi} e^{-j\frac{\pi}{2}} \cdot e^{j2\pi f_0 (1)t} + \frac{1}{3\pi} e^{-j\frac{\pi}{2}} \cdot e^{j2\pi f_0 (3)t} + \frac{1}{5\pi} e^{-j\frac{\pi}{2}} \cdot e^{j2\pi f_0 (5)t} =$$

$$= 0.5 + \frac{2}{\pi} \cos(2\pi f_0 t - \frac{\pi}{2}) + \frac{2}{3\pi} \cos(2\pi f_0 (3t - \frac{\pi}{2})) + \frac{2}{5\pi} \cos(2\pi f_0 (5t - \frac{\pi}{2}))$$



Ex. 3-10 : The same as Ex. 3-9, but use first 100 harmonics.
 just depict a plot of $x(t)$. You can observe so called
 Gibbs phenomenon, which occurs when there is some
 discontinuity in the signal.

Sol.:



From Fourier series to Fourier transform

- within Fourier analysis we compute $a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi f_0 k t} dt =$
 $= \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt = \frac{1}{2T_0} \int_{-T_0}^{T_0} x(t) e^{-jk\omega_0 t} dt$

How do obtain Fourier transform formula?

- let $T_0 \rightarrow \infty$ (signal need not to be periodic)
- compute not a_k , but $a_k \cdot 2 \cdot T_0$ (or simply don't consider $\frac{1}{2T_0}$)
- compute not for discrete radian frequencies $k\omega_0$, but for all ω .

Then we get Fourier transform $\hat{F}(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$

- problematic for power signals, some ω gives $F(\omega) \rightarrow \infty$

From Fourier transform to Short time Fourier transform (STFT)

- we multiply the signal with window function (some impulse) $w(t)$
- we obtain "picture" known as spectrogram (see below)

Continuous time: $STFT(\tau, \omega) = \int_{-\infty}^{\infty} x(t) \cdot w(t - \tau) e^{-j\omega t} dt$

Discrete time: $STFT[m, \omega] = \sum_{n=-\infty}^{\infty} x[n] \cdot w[n - m] \cdot e^{-j\omega n}$

window is
shifting through
all times

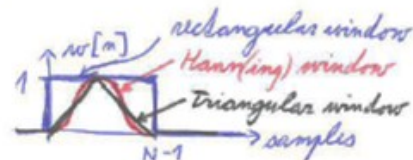
• Examples of window functions

1) Rectangular

$N \dots$ window width in samples

2) Triangular

3) Hanning window $w[n] = 0.5 \cdot (1 - \cos \frac{2\pi n}{N-1})$



• Resolution issues

- short window $\rightarrow$ good resolution of time, bad resolution of frequency
- long window $\rightarrow$ bad resolution of time, good resolution of frequency

103.5

Resolution issues - STFT in Matlab – chord A Major

```
% spectrograms with different Windows

%% initialize

clear;
close all;

%% defining parameters

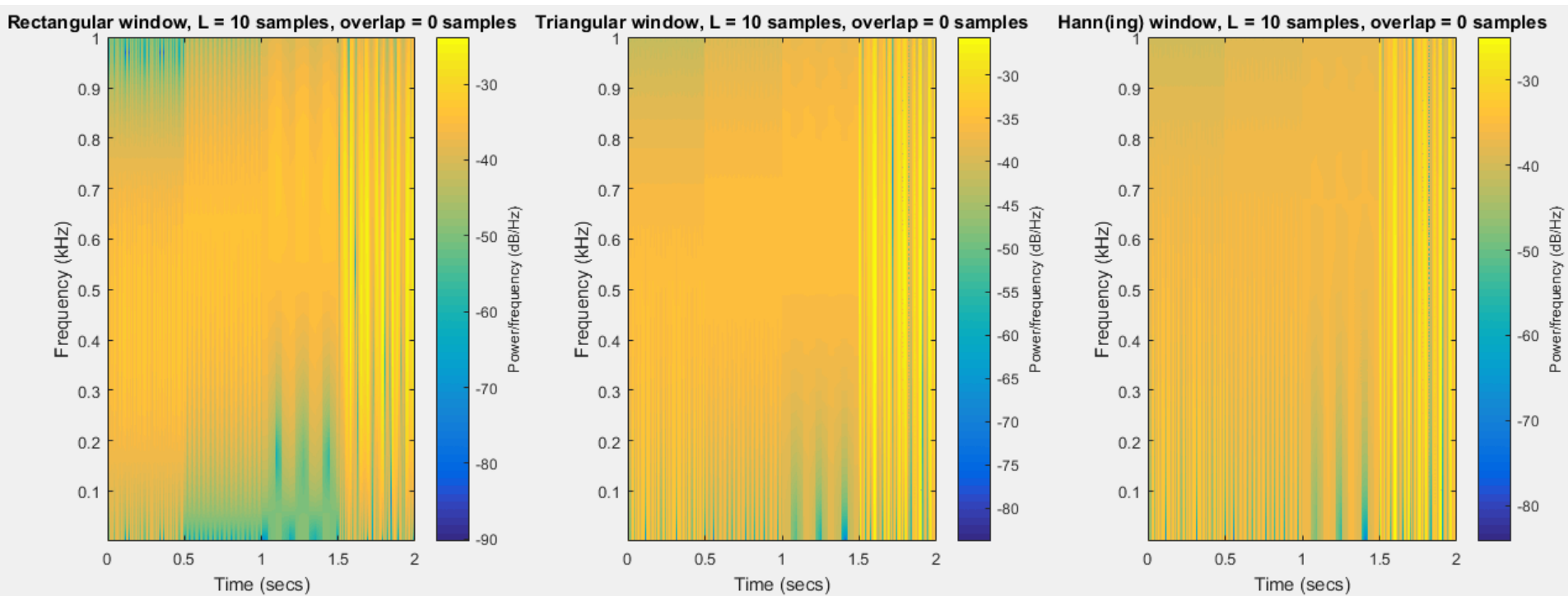
fs=6000; % sample frequency
tdur=0.5; %time duration of one tone in seconds
L=1000; %windows length
noverlap=0.9*L; %overlap samples

%% computation
Ts=1/fs;
t_tone=Ts:Ts:tdur;

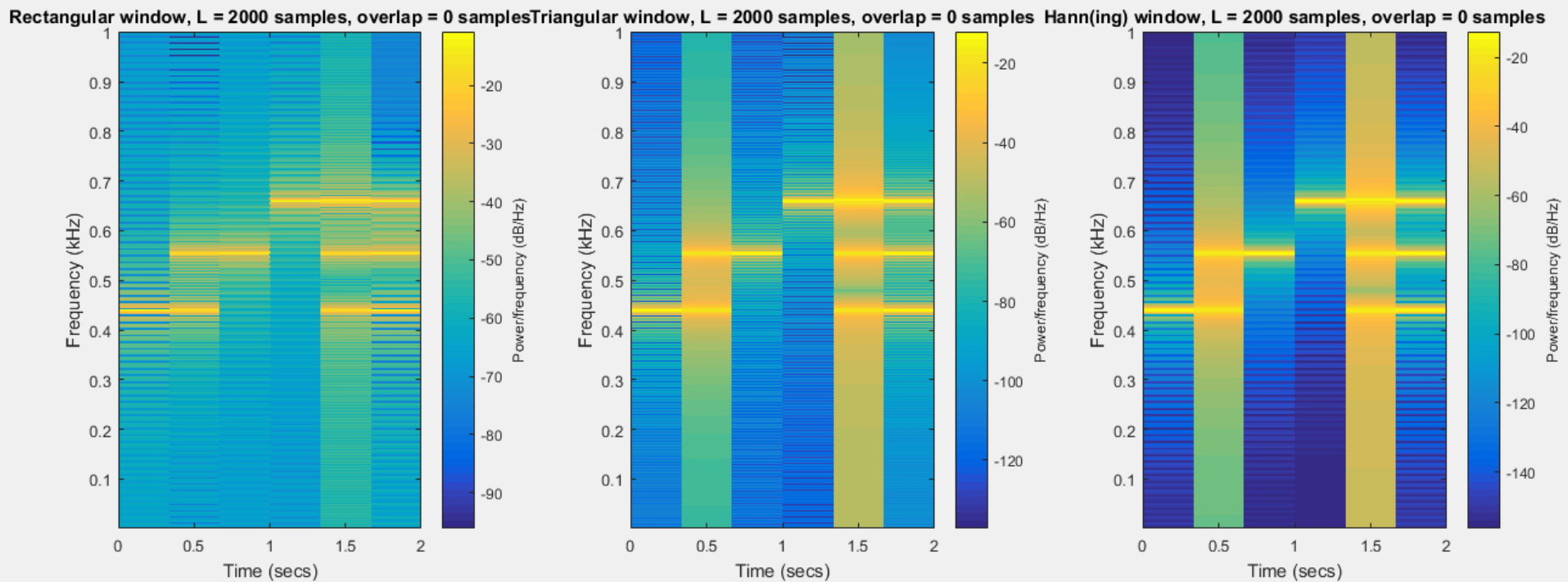
x1=sin(2*pi*440*t_tone); %tone A, f0=440 Hz, with fs=6000 ... 6000/440=13.6
samples/period
x2=sin(2*pi*440*2^(4/12)*t_tone); %tone C# (Cis), f0=554 Hz, with fs=6000 ... 10.8
samples/period
x3=sin(2*pi*440*2^(7/12)*t_tone); %tone E, f0=659 Hz, with fs=6000 ... 9.1
samples/period
x4=x1+x2+x3; %chord A Major (composed of tones A, C# and E), CZ: akord A Dur
x=[x1 x2 x3 x4];
sound(x,fs);
FigHandle = figure('Position', [100, 100, 1500, 500]); %defining position of corners
of the figure

subplot(1,3,1)
spectrogram(x,rectwin(L),noverlap,1:1000,fs,'yaxis');
title(sprintf('Rectangular window, L = %d samples, overlap = %d samples',L,noverlap))
subplot(1,3,2)
spectrogram(x,triang(L),noverlap,1:1000,fs,'yaxis')
title(sprintf('Triangular window, L = %d samples, overlap = %d samples',L,noverlap))
subplot(1,3,3)
spectrogram(x,hann(L),noverlap,1:1000,fs,'yaxis')
title(sprintf('Hann(ing) window, L = %d samples, overlap = %d samples',L,noverlap))
```

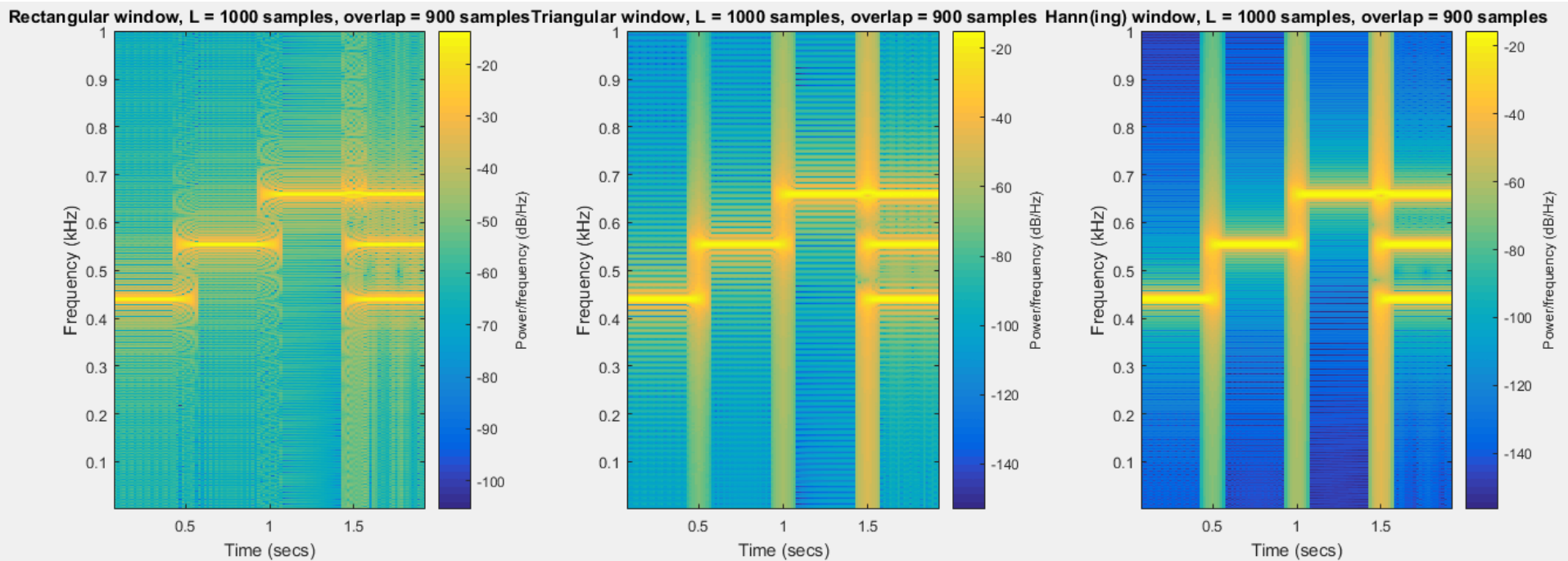

Good resolution of time, bad resolution of frequency, different windows.



Good resolution of frequency, bad resolution of time, different windows.



Sufficient resolution of both frequency and time, different windows.



Vocabulary EN/CZ

Beat note	Zázněj
Carrier frequency	Nosná frekvence
Greatest common divisor (g.c.d.)	Největší společný dělitel
Center frequency	Střední frekvence
Frequency deviation	Odchylka frekvence
Integer	Celé číslo
Base band	Základní pásmo
Broad band	Přeložené pásmo
Channel spacing	Odstup kanálů
Even	Sudý
Odd	Lichý
Fourier series	Fourierova řada
Window function	Jádro(vá funkce), okno

References

- McClellan, Schafer and Yoder, Signal Processing First, ISBN 0-13-065562-7., Prentice Hall, Upper Saddle River, NJ 07458. 2003 Pearson Education, Inc.

